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A new approach to the conformal invariance problem in quantum electrodynamics

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Abstract. A formulation of conformal-invariant quantum electrodynamics is suggested, in which new representations of the conformal group are used. This results in unusual laws for the electromagnetic potential and the transverse part of the current, and modified expressions are obtained for the vertex and three-point Green function. The program of the asymptotic charge calculation within the frames of conformal-invariant skeleton theory is formulated.

1. Formulation of the problem and the main result

It has been believed that the requirement of conformal invariance in electrodynamics results in a purely longitudinal photon propagator. Really, under the assumption that the electromagnetic potential $A_\mu(x)$ is transformed under the R-inversion $x_\mu \to x_\mu/x^2$ as a usual conformal vector† with a scale dimension $d_A = 1$,

$$A_\mu(x) \xrightarrow{R} U_R^A A_\mu(x) = x^{-2} g_{\mu\nu}(x) A_\nu(Rx) \tag{1}$$

where

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu/x^2, \qquad (Rx)_\mu = x_\mu/x^2,$$

a purely longitudinal expression $D^l_{\mu\nu}(x) \sim \partial_\mu \partial_\nu \ln x^2$ will be obtained for the propagator

$$D_{\mu\nu}(x_1, x_2) = \langle 0|A_\mu(x_1)A_\nu(x_2)|0\rangle.$$

Similarly, for the current $j_\mu(x)$ being considered as a conformal vector with a scale dimension $d_j = 3$ the transformation law

$$j_\mu(x) \xrightarrow{R} U_R^j j_\mu(x) = (x^2)^{-3} g_{\mu\nu}(x) j_\nu(Rx) \tag{2}$$

is obtained, that leads to the transverse propagator.

Up to recent times the fact of longitudinality was considered to be the main obstacle in the attempts to formulate the conformal-invariant electrodynamics (see e.g. the review by Todorov *et al* (1978) and references therein). A new formulation of Euclidean conformally-invariant electrodynamics is suggested in the present paper, where another transformation law is accepted instead of (1) and (2) (its derivation is

† A review of the present status and the methods of the conformally-invariant field theory is given in the reviews by Todorov *et al* (1978) and Fradkin and Palchik (1978).

given below, see also Kozhevnikov *et al* (1981) and Fradkin *et al* (1982)):

$$A_\mu(x) \xrightarrow{R} \tilde{U}_R^A A_\mu(x) = [(1-P)U_R^A(1-P) + U_R^A P]A_\mu(x), \tag{3}$$

$$j_\mu(x) \xrightarrow{R} \tilde{U}_R^j j_\mu(x) = [U_R^j(1-P) + PU_R^j P]j_\mu(x), \tag{4}$$

where $P = \partial_\mu\partial_\nu/\Box$ is a longitudinal projection operator in Euclidean space. The presence of a longitudinal part in the current j_μ is caused by the fact that it is considered as a quantum field in Euclidean space, see (8) and (9*a*). With the transformation law (3) for the propagator of the field $A_\mu(x)$ we obtain the expression containing a transverse part (in generalised Feynman gauge, see (9*a*)):

$$D_{\mu\nu}(x) = \frac{1}{4\pi^2}\left[\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\frac{1}{x^2} - \eta\frac{\partial_\mu\partial_\nu}{\Box}\frac{1}{x^2}\right] \tag{5}$$

where η is a gauge parameter.

2. New representation of the conformal group for the field A_μ and j_μ

Let us reveal the source of the above difficulty. In our opinion it consists of the fact that the conformal group representations given by formulae (1) and (2) are indecomposable (Dobrev *et al* 1977). In the space of each of these representations there is an invariant subspace. Let us consider, for example, (1). Let H be the space of all vector fields A_μ. It can easily be seen that the invariant subspace denoted by H^l consists of longitudinal functions. If $A_\mu^l = \partial_\mu\varphi(x)$ then the transformed function $U_R^A A_\mu^l(x) = x^{-2}g_{\mu\nu}(x)A_\nu^l(Rx)$ is also longitudinal. It is significant that the complement H^{tr} to this subspace consisting of transverse functions is not invariant.

Let us consider now the representation (3). Let g be any conformal transformation. As is known, it can be represented as a combination of Lorentz transformations, translations and R-inversions. Let U_g be an indecomposable representation corresponding to the transformation law (1). We now define a new representation $\tilde{U}_g$ similarly to (3):

$$\tilde{U}_g = (1-P)U_g(1-P) + U_g P \tag{6}$$

For the operators $\tilde{U}_g$ we find

$$\begin{aligned}\tilde{U}_{g_1}\tilde{U}_{g_2} &= (1-P)U_{g_1}(1-P)^2U_{g_2}(1-P) + U_{g_1}PU_{g_2}P \\ &\quad + (1-P)U_{g_1}U_{g_2}P + U_{g_1}P(1-P)U_{g_2}(1-P) \\ &= (1-P)U_{g_1}U_{g_2}(1-P) + U_{g_1}U_{g_2}P.\end{aligned}$$

While proving the latter equality it was taken into account that due to invariance of the subspace H^l we have $PU_gP = U_gP$, $(1-P)U_gP = 0$ and also $P^2 = P$. Since $U_{g_1}U_{g_2} = U_{g_1g_2}$ we obtain that the operators (6) satisfy the group law $\tilde{U}_{g_1}\tilde{U}_{g_2} = \tilde{U}_{g_1g_2}$ as well. It can easily be seen that the new representation $\tilde{U}_g$ is reducible. It is a direct sum of two irreducible representations; one of them acts in the subspace of longitudinal functions $A_\mu^l \in H^l$, and the other in the subspace of transverse functions $A_\mu^{\mathrm{tr}} \in H^{\mathrm{tr}}$. Note that the latter representation is equivalent to the irreducible representation induced by the indecomposable representation U_g in a quotient space $H/H^l \sim H^{\mathrm{tr}}$.

Now we consider the transformation law (2). In this case the invariant subspace consists of transverse functions j_μ^{tr}: if $\partial_\mu j_\mu^{\mathrm{tr}}=0$, then it is easy to verify that $\partial_\mu j_\mu^{\mathrm{tr}'}=0$, where $j_\mu^{\mathrm{tr}'}(x)=(x^2)^{-3}g_{\mu\nu}(x)j_\nu^{\mathrm{tr}}(Rx)$. All considerations concerning A_μ are repeated with the obvious changes for $j_\mu(x)$, which results in the transformation law (4).

3. Modified expression for the vertex and three-point Green function

It can be shown that the new transformation law leads to the following expressions for three-point Green functions (in Euclidean space):

$$\begin{aligned}G_\mu(x_1|x_2x_3)&=\langle 0|A_\mu(x_1)\psi(x_2)\bar\psi(x_3)|0\rangle\\&=g(\delta_{\mu\nu}-\partial_\mu^{x_1}\partial_\nu^{x_1}/\Box_{x_1})C_\nu(x_1|x_2x_3)+C_\mu^l(x_1|x_2x_3)\end{aligned}\tag{7}$$

where g is a coupling constant,

$$\begin{aligned}\Gamma_\mu(x_1|x_2x_3)&=\langle 0|j_\mu(x_1)\psi(x_2)\bar\psi(x_3)|0\rangle\\&=g\tilde C_\mu^{\mathrm{tr}}(x_1|x_2x_3)+(\partial_\mu^{x_1}\partial_\nu^{x_1}/\Box_{x_1})\tilde C_\nu(x_1|x_2x_3).\end{aligned}\tag{8}$$

The projection operators $(\delta_{\mu\nu}-\partial_\mu\partial_\nu/\Box)$ and $\partial_\mu\partial_\nu/\Box$ entering here reflect the structure of transformation laws (3) and (4). The functions C_μ, C_μ^l and $\tilde C_\mu^{\mathrm{tr}}$, $\tilde C_\mu$ are invariant with respect to the old transformation laws (1) and (2). They are (Todorov *et al* 1978, Fradkin and Palchik 1978)

$$C_\mu(x_1|x_2x_3)=\frac{1}{4}\frac{1}{(x_{23}^2)^{d-\frac{1}{2}}}\frac{\hat x_{21}}{x_{12}^2}\gamma_\mu\frac{\hat x_{13}}{x_{12}^2},$$

$$C_\mu^l(x_1|x_2x_3)=\frac{\eta e}{(4\pi)^2}\partial_\mu^{x_1}\left(\ln\frac{x_{13}^2}{x_{12}^2}\frac{\hat x_{23}}{(x_{23}^2)^{d+\frac{1}{2}}}\right),$$

$$\begin{aligned}\tilde C_\mu^{\mathrm{tr}}(x_1|x_2x_3)&=\frac{\hat x_{21}}{(x_{12}^2)^2}\gamma_\mu\frac{\hat x_{13}}{(x_{13}^2)^2}\frac{1}{(x_{23}^2)^{d-\frac{3}{2}}}\\&\quad-\frac{\hat x_{23}}{(x_{23}^2)^{d-\frac{1}{2}}}\frac{1}{x_{13}^2}\frac{1}{x_{12}^2}\left(\frac{(x_{21})_\mu}{x_{12}^2}-\frac{(x_{31})_\mu}{x_{13}^2}\right),\end{aligned}$$

$$\tilde C_\mu(x_1|x_2x_3)=-\frac{e}{2\pi^2}\frac{\hat x_{23}}{(x_{23}^2)^{d-\frac{1}{2}}}\frac{1}{x_{12}^2}\frac{1}{x_{13}^2}\left(\frac{(x_{21})_\mu}{x_{12}^2}-\frac{(x_{31})_\mu}{x_{13}^2}\right),$$

where d is a scale dimension of the spinor field. Here the normalising factors are chosen from the condition

$$G_\mu(x_1|x_2x_3)=\int dx_4\,D_{\mu\nu}(x_{14})\Gamma_\nu(x_4|x_2x_3)\tag{9}$$

where $D_{\mu\nu}(x)$ is the photon propagator (5), and normalisations of longitudinal parts are fixed by Ward identities

$$\begin{aligned}&\eta^{-1}\Box\partial_\nu D_{\mu\nu}(x)=\partial_\mu\delta(x),\\&\eta^{-1}\Box_{x_1}\partial_\mu^{x_1}G_\mu(x_1|x_2x_3)=e[\delta(x_{12})-\delta(x_{13})]G(x_{23}),\end{aligned}\tag{9a}$$

where e is the asymptotic electric charge, and

$$G(x_2x_3)=\langle 0|\psi(x_2)\bar\psi(x_3)|0\rangle=\hat x_{23}/(x_{23}^2)^{d+\frac{1}{2}}\tag{10}$$

The expressions (7) and (8) can be obtained as follows. If one returns to the old transformation law (1) and (2), there are, as has already been said, two independent invariant functions. For definiteness consider the Green functions with a current. Then the functions $\tilde{C}_\mu^{\rm tr}$ and $\tilde{C}_\mu$ turn out to be such Green functions. It is easy to see that $\tilde{C}_\mu^{\rm tr}$ satisfies both the transformation laws (2) and (4) because in this case the second term in (4) is absent due to the transversality of the function $\tilde{C}_\mu^{\rm tr}$ and in the first term one can omit the projector $(1-P)$. Another independent function invariant under (4) has been determined as the longitudinal part $(\partial_\mu^{x_1}\partial_\nu^{x_1}/\Box_{x_1})\tilde{C}_\nu(x_1|x_2x_3)=\tilde{C}_\mu^{\rm long}(x_1|x_2x_3)$. To verify this, substitute it into (4). The first term in the RHS disappears due to its longitudinality. The second term results in the equation

$$C_\mu^{\rm long}(x_1|x_2x_3)=(\cdots)\frac{\partial_\mu^{x_1}\partial_\rho^{x_1}}{\Box_{x_1}}\frac{1}{(x_1^2)^3}g_{\rho\sigma}(x_1)\frac{\partial_\sigma^{Rx_1}\partial_\tau^{Rx_1}}{\Box_{Rx_1}}C_\tau^{\rm long}(Rx_1|Rx_2Rx_3) \tag{4a}$$

where $(\cdots)$ is a set of multipliers which arise as a result of the transformation of the spinor indices (see e.g. Todorov *et al* 1978 or Fradkin and Palchik 1978). Allowing for the relations

$$\partial_\mu^x(x^2)^{-3}g_{\mu\nu}(x)=0, \qquad \partial_\mu^x=x^{-2}g_{\mu\nu}(x)\partial_\nu^{Rx},$$

the RHS transforms as

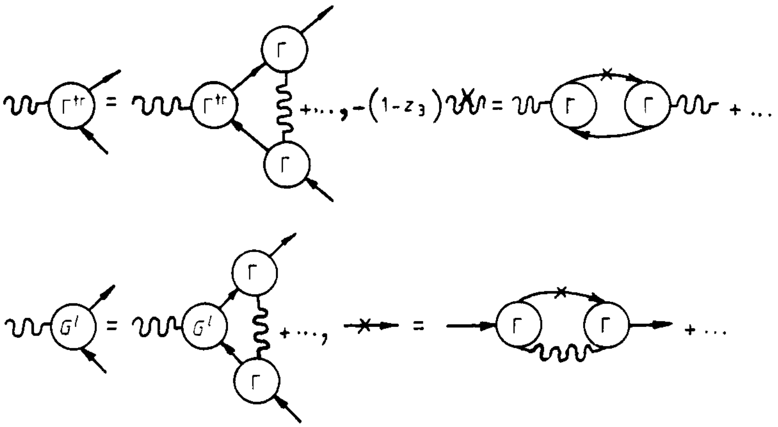
$$\begin{aligned}
&(\partial_\mu^{x_1}/\Box_{x_1})(x^2)^{-3}g_{\rho\sigma}(x_1)\partial_\rho^{x_1}\partial_\sigma^{Rx_1}\frac{\partial_\tau^{Rx_1}}{\Box_{Rx_1}}\tilde{C}_\tau(Rx_1|Rx_2Rx_3)\\
&\quad=(\partial_\mu^{x_1}/\Box_{x_1})(x_1^2)^{-4}\partial_\tau^{Rx_1}\tilde{C}_\tau(Rx_1|Rx_2Rx_3)\\
&\quad=(\partial_\mu^{x_1}\partial_\nu^{x_1}/\Box_{x_1})(x_1^2)^{-3}g_{\nu\tau}(x_1)\tilde{C}_\tau(Rx_1|Rx_2Rx_3)\\
&\quad=(\partial_\mu^{x_1}\partial_\nu^{x_1}/\Box_{x_1})\tilde{C}_\nu(x_1/x_2x_3)=C_\mu^{\rm long}(x_1|x_2x_3).
\end{aligned}$$

Thus (4*a*) is proved and (8) is the most general expression satisfying the law (4). The relation (7) is being proved in an analogous manner.

4. The program of the asymptotic charge calculation

Let us consider the scheme of asymptotic charge calculation. Following Johnson *et al* (1967), we assume that renormalised photon propagator in the limit of large momenta coincides with a free propagator. As is shown by Baker and Johnson (1969), this solution can be realised only for the charge value which is a zero of the Gell-Mann–Low function. As is known, such a solution should have the asymptotic conformal symmetry. The conformally-invariant QED considered thus coincides with real theory in the large momenta range, and the asymptotic charge value can be obtained by calculating the normalisation of the quantity $G^{-1}\Gamma_\mu$ where G and Γ_μ are the conformal-invariant propagator and 'vertex' (10) and (8), respectively.

The required normalisations can be calculated from skeleton equations for the vertex Γ_μ (or Green function G_μ) and fermion and photon propagators G and $D_{\mu\nu}$, respectively. Note that since the Green function has two independent structures, the vertex equation is equivalent to two equations for the parameters entering the theory. In practice, it is easy to write the vertex equation as two equations for the conformally-invariant functions $\Gamma_\mu^{\rm tr}=g\tilde{C}_\mu^{\rm tr}$ and $G_\mu^l=C_\mu^l$ determined above. As a result, we have in the three-vertex approximation:



We used here standard notations (see e.g. Fradkin and Palchik 1978). However, of these four equations only three are independent, because two equations (13) are equivalent due to the Ward identity (10). Conformal invariance of these equations with respect to transformations (3) and (4) is provided by the invariance of all vertices and propagators entering them (see Kozhevnikov *et al* (1981) for detail).

Let us count the number of independent parameters to be determined. They are the scale dimension d of the spinor field, the coupling constant g, the renormalisation constant z_3 and the gauge parameter η. There are three equations for their determination: two equations (12) and one of equations (13). The parameter η remains free, thus reflecting the gauge freedom.

Therefore, parameters g and d and, hence, the asymptotic charge, expressed through their gauge independent combination, can be calculated from (12) and (13).

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